rights less than any assignable part of a right. Therefore the four angles together of this stable and given quadrilateral either are equal to four rights or greater.

But then (from P. XVI.) is established the hypothesis either of right angle or of obtuse angle ; and therefore (from P. V. and P. VI.) the hypothesis of acute angle is destroyed.

So is established that the hypothesis of acute angle will be destroyed, if two straights existing in the same plane so approach each other mutually ever more, that nevertheless their distance is always greater than any assigned length.

Hoc antem erat demonstrandum.

COROLLARY I. But (the hypothesis of acute angle destroyed) the controverted Pronunciatum Euclidaeum is manifest from P. 13 of this ; just as that by me in this place it would be disclosed I promised in Scholion III after P. XXI of this, where we spoke of the attempt of the Arab Nassaradin.

COROLLARY II. On the other hand from this proposition, and from the preceding XXIII is manifestly gathered that not sufficient for establishing Euclidean geometry are two following points. One is : that we designate by the name of parallels those straights, which existing in the same plane possess a common perpendicular. The second indeed ; that all straights existing in the same plane, of which there is no common perpendicular, and therefore which according to the assumed definition are not parallel, must, being produced toward either part ever more, somewhere meet each other, if not at a finite, at least at an infinite distance.

For again it would be requisite to demonstrate, that any two straights existing in the same plane, upon which a certain straight cutting makes two internal angles toward the same parts less than two right angles, nowhere else can receive a common perpendicular.

But that, this demonstrated, Euclidean geometry is most exactly established, will be shown below.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

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[Continued from January Number.]

XXVII. Let ABC be a triangle, right-angled at C. With O, the middle of AB, as a center, describe a circle to which either of the other sides, as BC, shall be tangent. Then,

$$BD \cdot BE = \overline{BP}^2;$$

or $(\frac{1}{2}c - \frac{1}{2}b)(\frac{1}{2}c + \frac{1}{2}b) = \frac{1}{4}a^2$. $\therefore c^2 = a^2 + b^2$.

This and XVI are special cases of a more general form. For O may be any point in AB, such that the ratio of OB to AB shall be n. Our equation would then become $(nc - nb)(nc + nb) = n^2a^2$; whence, $c^2 = a^2 + b^2$.

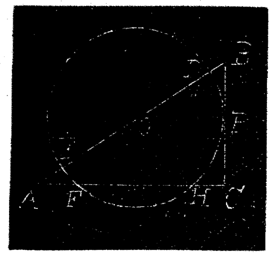

Fig. 21.

XXVIII. Fig. 21.

Suppose $BC < AC$. Then $HC \cdot FC = \overline{PC}^2$.

But $HC \cdot FC = AF \cdot AH = AE \cdot AD = (\frac{1}{2}c - \frac{1}{2}b)(\frac{1}{2}c + \frac{1}{2}b)$, and $\overline{PC}^2 = \frac{1}{4}a^2$.

$\therefore c^2 = a^2 + b^2$.

From $BC < AC$ pass to $BC = AC$ by the theory of limits.

XXIX. Let ABC be a triangle, right-angled at C. Describe a circle, such that its center O shall be in AB, and to which the other sides shall be tangent.

Draw OD perpendicular to AB. Then,

$$\overline{AT}^2 = AE \cdot AF = \overline{AO}^2 - \overline{EO}^2 = \overline{AO}^2 - \overline{TC}^2 \quad \ldots\ldots\ldots\ldots \quad (1).$$

$$\overline{BP}^2 = BF \cdot BE = \overline{BO}^2 - \overline{FO}^2 = \overline{BO}^2 - \overline{CP}^2 \quad \ldots\ldots\ldots\ldots \quad (2).$$

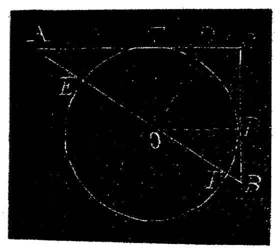

Fig. 22.

Now, $AO : OT :: AD : OD$;

$\therefore AO \cdot OD = OT \cdot AD$.

And, since $OD = OB$, $OT = TC = CP$, and $AD = AT + TD = AT + BP$,

$$\therefore AT \cdot TC + CP \cdot BP = AO \cdot OB \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

Adding (1), (2), and $2 \times (3)$,

$$\overline{AT}^2 + \overline{BP}^2 + 2AT \cdot TC + 2CP \cdot BP = \overline{AO}^2 - \overline{TC}^2 + \overline{BO}^2 - \overline{CP}^2 + 2AO \cdot OB;$$

$$\therefore \overline{AT}^2 + 2AT \cdot TC + \overline{TC}^2 + \overline{BP}^2 + 2BP \cdot CP + \overline{CP}^2 = \overline{AO}^2 + 2AO \cdot OB + \overline{BO}^2;$$

$$\therefore \overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2.$$

XXX. Let ABC be a triangle, right-angled at C. Describe a circle, such that its center O shall be in one of the legs, as AC, and to which the other leg

and hypotenuse shall be tangent.

Then $\overline{AD}^2=AE\cdot AC=\overline{AC}^2-2AC\cdot OE$;

and $\overline{BD}^2=\overline{BC}^2$;

$\therefore$ Adding, $\overline{AD}^2+\overline{BD}^2=\overline{AC}^2+\overline{BC}^2-2AC\cdot OE$.

$\therefore\ \overline{AD}^2+2AC\cdot OE+\overline{BD}^2=\overline{AC}^2+\overline{BC}^2$.

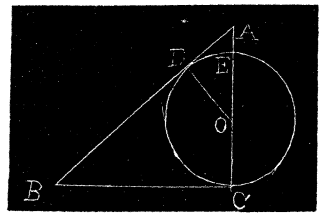

Fig. 23.

Now $AC : AC :: OD(=OE) : BC(=BD)$;

$\therefore\ AD\cdot BD=AC\cdot OE$; $\therefore\ \overline{AD}^2+2AD\cdot BD+\overline{BD}^2=\overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

XXXI. Fig. 23.

$\overline{AD}^2(=(AB-BD)^2)=\overline{AC}^2-2AC\cdot OE$,

$\therefore\ \overline{AB}^2-2AB\cdot BD+\overline{BD}^2=\overline{AC}^2-2AC\cdot OE$.

Adding $\overline{BD}^2=\overline{BC}^2$, $\overline{AB}^2-2BD\cdot AD=\overline{AC}^2+\overline{BC}^2-2AC\cdot OE$.

$\therefore\ \overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

XXXII. Let ABC be a triangle right-angled at C. Draw AE parallel to BC and $=AC$. With O, the middle of AE, as a center, describe a circle, to which both AC and BC shall be tangent. Then,

$\overline{BT}^2(=(a-b)^2)=BD\cdot BA=c(c-AD)$ . . . . (1).

Also, $AD : a :: 2b : c$.

$\therefore\ AD=2ab/c$ . . . . (2).

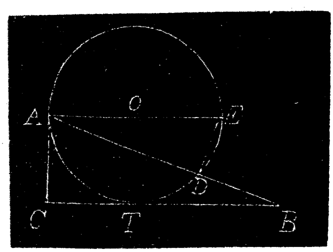

Fig. 24.

(2) in (1), $(a-b)^2=c^2-2ab$. $\therefore\ c^2=a^2+b^2$.

From unequal sides about the right angle pass to equal sides by the theory of limits.

[To be Continued.]